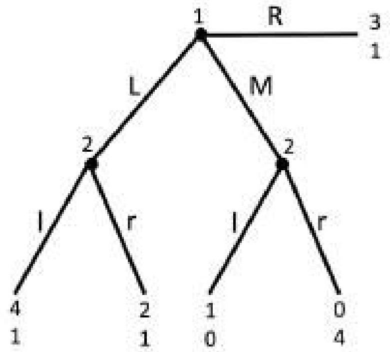
Exercise 1 - 30%

1. Find *all* (pure and mixed) Nash equilibria of the following game:

	L	R
T	5,2	0,0
B	3,0	3,2

2. For the extensive form game below, answer the following questions:



   (a) How many subgames are there in this game (excluding the game itself)?

   (b) Write down the strategies of the two players.

   (c) Find all pure strategy subgame perfect Nash equilibria.

   (d) Write down the normal-form ("matrix-form") of the game and find all pure strategy Nash equilibria.

3. Consider the finitely repeated game where the stage game shown below is played $T < \infty$ times and payoffs are the sum of payoffs from each stage (no discounting).

	L	R
T	11,-1	0,0
B	5,5	-2,6

   (a) Suppose $T = 2$. Does there exist a subgame perfect Nash equilibrium such that (B,L) is played in the first stage?

   (b) Does it change your answer to the question above if we let $T > 2$?

Answer# 1:

Given the game in matrix form as:

L	R
5, 2	0, 0
3, 0	3, 2

According to the game, the Row’s problem would be:

Maximize p_t [p_l * 5 + p_r * 3] + p_b [p_l * 0 + p_r * 3]

(p_t, p_b)   such that $p_t + p_b = 1$;

$p_t \geq 0$; and $p_b \geq 0$.

Here (p_t, p_b) be the probabilities with which Row plays the Top and Bottom respectively (not shown in this figure) and (p_l, p_r) be the probabilities with which Row plays Left and Right respectively.

Let λ, μ_t, and μ_b be the Kuhn – Tucker multipliers on the constraints, so we can write the Lagrangian equation as:

$$Z = 5\, p_t\, p_l + 3\, p_t\, p_r + 3\, p_b\, p_r - \lambda(p_t + p_b - 1) - \mu_t\, p_t - \mu_b\, p_b$$

The required first – order Kuhn – Tucker conditions are:

$\partial Z / \partial p_t = 5p_l + 3p_r - (\lambda + \mu_t) = 0$ ………………………… (1)

$\partial Z / \partial p_b = 3p_r - (\lambda + \mu_b) = 0$ ……………………………… (2)

For pure strategy solutions, we only consider $p_t, p_b > 0$. The complementary slackness conditions then imply $\mu_t = \mu_b = 0$. Using the fact that $p_t + p_b = 1$, we can state $p_l + p_r = 1$.

From equation: (2), it is evident that $p_r = 0$, hence $p_l = 1$.

Following the same procedure for Column, we find that $p_t = 0$, $p_b = 1$.

Hence the payoff of the game is 1 and the strategies would be: (p_l, p_r) = (1, 0) and

(p_t, p_b) = (0, 1) [Nash equilibrium solution].

Answer# 2:

(a) As per the diagram, there are are two subgames: the game as a **whole** and the part of the game beginning with L and M playing ‘l’ and ‘r’ strategies at two decision nodes. If we exclude the game itself, then there is *one* subgame.

(b) The strategies of two players would be:

L	M
4, 1	1, 0
2, 1	0, 4

Figure: 1

The extensive form of the strategy of player: 2 would be (in case he wants to play R and / or M):

3, 1	3, 1
1, 0	0, 4

Figure: 2

Figure: 1 depicts the scenario when two players start playing from the *root* (termed as 1) and player: 1 plays L and from the decision node can play either l (left) or r (right) with payoffs (4,1) and (2, 1) respectively. Whereas player: 2 opts for playing M and *if* he decides to play M then depending on the decision node can play either l or r yielding payoffs (1, 0) and (0, 4) respectively.

Now if player: 2 opts to play R from root node# 1, then his payoff would be (3, 1), (3, 1) depending on strategies to play R and M after entering the subgame with player: 1 who opts to play L with strategies l and r respectively. Player: 2 can opt for playing R as its payoff is higher compared to playing M.

(c) For figure: 1, proceeding like what we did for part: a we directly write the objective function as (for Row maximization):

$$Y = 4p_t\, p_l + 2p_t\, p_r + 3\, p_b\, p_l – \lambda(p_t + p_b – 1) - \mu_t\, p_t – \mu_b\, p_b$$

The first order conditions and the restrictive conditions on slack variables give us the two equations as:

$$4\ p_l + 2\ p_r = p_l$$

$$\text{or, } 3\ p_l + 2\ p_r = 0$$

And we have: $p_l + p_r = 1$. By solving the two equations we get:

$$p_l = -\ 2 / 3,\ p_r = 1$$

Proceeding likewise, for Column maximization we get:

$$p_t = 1,\ p_b = -\ 2 / 3.$$

Hence the strategies would be $(p_t, p_b) = (1, -\ 2 / 3)$ and $(p_l, p_r) = (-\ 2 / 3, 1)$ - which is the SPNE solution.

Similarly for figure: 2 we have the objective function as (for Row maximization):

$$V = 3p_t\ p_l + p_t\ p_r + 3p_b\ p_l - \lambda(p_t + p_b - 1) - \mu_t\ p_t - \mu_b\ p_b$$

The first order conditions and the restrictive conditions on slack variables give us the two equations as:

$$3p_l + p_r = 3p_l$$

$$\text{or, } p_r = 0 \text{ and } p_l = 1$$

Similarly for Column maximization, we get: $p_t = 1$ and $p_b = 0$.

$(p_t, p_b) = (1, 0)$ and $(p_l, p_r) = (0, 1)$ – which is another SPNE solution.

(d)

4, 1	1, 0
2, 1	0, 4

3, 1	3, 1
1, 0	0, 4

Figure: 3

Figure: 3 is the extensive form of the game. In order to find the pure strategy Nash Equilibrium, we have to assume a positive probability σ which would be attached to both L and M such that

$$4\sigma + 2(1 - \sigma) = \sigma + 0 * (1 - \sigma);$$

$$\sigma + (1 - \sigma) = 0 + 4(1 - \sigma);$$

$$\sigma + 0 * (1 - \sigma) = 3\sigma + (1 - \sigma); \text{ and}$$

$$0 + 4 * (1 - \sigma) = 3\sigma + (1 - \sigma).$$

Solving each of the equations gives us σ = -2, 3 / 4, -1 and 3 / 2.

The *only* feasible value of σ is 3 / 4. Hence the pure strategy NE solution is (3 /4, 1 / 4) for the two players.

Answer# 3:

(a) Given the game as:

L	R
11, -1	0, 0
5, 5	-2, 6

(a) In order to solve the game, objective function as (for Row maximization):

$$ł = 11p_t\, p_l + 5p_t\, p_r - p_b\, p_l + 5p_b\, p_r - \lambda(p_t + p_b - 1) - \mu_t\, p_t - \mu_b\, p_b$$

The first order conditions and the restrictive conditions on slack variables give us the two equations as:

$$11p_l - 5p_r = - p_l + 5p_r$$

or, $12p_l - 10p_r = 0$

And we have: $p_l + p_r = 1$. By solving the two equations we get:

$$p_l = 5 / 11,\ p_r = 6 / 11.$$

Now similarly by Column maximization, we get:

$$p_t = 6 / 11,\ p_b = 5 / 11.$$

Now the payoff (v) = $11p_t\, p_l + 5p_t\, p_r - p_b\, p_l + 5p_b\, p_r = 630 / 121$

Hence player: 1's share = (6 / 11) * 630 / 121 = 2.83 (approximately)

And player: 2's share = (5 / 11) * 630 / 121 = 2.36 (approximately).

Hence it may be seen that both the players share almost equal share of the total payoffs.

However at (B, L) = (5, 5), there might be a SPNE where the share of the payoffs may be equally divided and each player gets a payoff in the amount of [(1 / 2) * (630 / 121)] = 2.60 (approximately) and the game ends there *if* both the players agree [T = 2].

(b) At T = 1, by assumption, player: 1 begins the game and offers a share of the total payoff to player: 2. If player: 2 accepts the offer, the game ends there. In case of a discount factor, from T = 2 onwards, both the players payoffs start getting discounted. However, according to the problem, since there is no discounting, the game will continue only if one of the players rejects the offer made by the other (for T > 2). However, in case of equal payoffs, both the players at any stage might agree to end the game as none of them faces an economic loss.

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