Exercise 2 - 30%

Two neighbors are taking care of a common road leading to their villas. Each of them exert an effort $e_i \geq 0, i = 1, 2$. The resulting quality of the road is

$$f(e_1, e_2) = a_1 e_1 + a_2 e_2 - e_1 e_2,$$

where $a_1, a_2 > 0$ are constants such that $2a_1 - a_2, 2a_2 - a_1 \geq 0$.
Exerting effort is costly. More precisely, each neighbor has a quadratic cost of effort:

$$C_i(e_i) = e_i^2, i = 1, 2.$$

The payoff of neighbor i, U_i, is equal to the quality of the road minus his cost of effort.

(a) Suppose the neighbors choose their effort levels simultaneously and independently. Derive the best response functions. Find the pure strategy Nash equilibrium of this game.

For the rest of the questions, assume that $a_1 = a_2 = 1$.

(b) Calculate the payoffs of the neighbors in the Nash equilibrium.

(c) Find the aggregate effort level $\bar{e} = \bar{e}_1 + \bar{e}_2$ that maximizes the sum of the neighbors payoffs. Calculate the corresponding payoffs of the neighbors, assuming that they contribute equally ($e_1 = e_2$).

(d) Suppose the interaction between the neighbors studied in (a) is repeated over an infinite time horizon $t = 1, 2, ..., \infty$. Assume that the neighbors discount future payoffs with the discount factor $\delta \in (0, 1)$ and that they maximize the sum of discounted payoffs. Suggest strategies in this infinitely repeated game that yields average discounted payoffs equal to the neighbors' payoffs obtained in (c). Find the minimal discount factor δ such that the strategies constitute a subgame perfect Nash equilibrium.

Answer:

(a) According to the problem, the quality of the road function is given as:

$f(e_1, e_2) = a_1 e_1 + a_2 e_2 - e_1 e_2$ $(a_1, a_2 > 0)$

Cost of effort function: $C_i(e_i) = e_i^2$ $(i = 1, 2)$

And the Payoff $(U_i) = f(e_1, e_2) - C_i(e_i) = a_1 e_1 + a_2 e_2 - e_1 e_2 - e_i^2$

Also given the constraints as $2a_1 – a_2 \geq 0$, and $2a_2 – a_1 \geq 0$

Hence the problem is to maximize the payoff function by forming the Lagrangian equation as (in case of *simultaneous* actions):

Maximize $\Psi = a_1 e_1 + a_2 e_2 - e_1 e_2 - e_1^2 - e_2^2 + \lambda_1 [2a_1 - a_2 - \beta] + \lambda_2 [2a_2 - a_1 - \gamma]$

subject to e_1, e_2, λ_1, and λ_2 where 'β' and 'γ' are **surplus** variables ($\lambda_1, \lambda_2, \beta, \gamma \geq 0$).

The required first – order conditions are:

$\partial\Psi / \partial e_1 = a_1 - e_2 - 2e_1 = 0$ …………………… (1)

$\partial\Psi / \partial e_2 = a_2 - e_1 - 2e_2 = 0$ ………………….... (2)

$\partial \Psi / \partial\lambda_1 = 2a_1 - a_2 - \beta = 0$ …………………….. (3)

$\partial\Psi / \partial\lambda_2 = 2a_2 - a_1 - \gamma = 0$ …………………….. (4)

Solving equations: (3) and (4) we get:

$a_1 = (2\beta + \gamma) / 3$; and $a_2 = (\beta + 2\gamma) / 3$.

Now substituting the values of a_1 and a_2 in equations: (1) and (2) we get:

$2e_1 + e_2 = a_1 = (2\beta + \gamma) / 3$ ………………………. (5)

and

$e_1 + 2e_2 = a_2 = (\beta + 2\gamma) / 3$ ………………………. (6)

By solving equations: (5) and (6) we get the pure strategy Nash equilibrium as:

$$(e_1^*, e_2^*) = (\beta / 3, \gamma / 3) = (1 / 3) * [(2a_1 - a_2), (2a_2 - a_1)]$$

Here one important factor should be noted. When the value of the surplus variables become zero, the Nash equilibrium (NE) solution becomes zero (non – existent) and when they are infinitesimal ($\beta, \gamma \rightarrow \infty$), then we can write:

$2a_1 - a_2 = 0 = a_1 - 2a_2$ – which *again* gives us the non – existent NE solution.

Thus the pure strategy NE solution *exists* when $2a_1 - a_2 \neq 0$ and $a_1 - 2a_2 \neq 0$.

The best response function for neighbor: 1 would be **either**:

$e_1 = (a_1 - e_2) / 2$ *or* $e_1 = a_2 - 2e_2$

The best response function for neighbor: 2 would be **either**:

$e_2 = (a_1 - 2e_1) / 2$ *or* $e_2 = a_1 - 2e_1$

Now when the neighbors act *independently* then we may write the **individual** utility functions for neighbor: 1 and neighbor: 2 separately as:

$U_1 = a_1 e_1 + a_2 e_2 - e_1 e_2 - e_1^2$ and $U_2 = a_1 e_1 + a_2 e_2 - e_1 e_2 - e_2^2$

The necessary Lagrangian equations for neighbor: 1 would be:

Maximize $\Omega = a_1 e_1 + a_2 e_2 - e_1 e_2 - e_1^2 + \lambda_1 [2a_1 - a_2 - \beta] + \lambda_2 [2a_2 - a_1 - \gamma]$
subject to e_1, e_2, λ_1 and λ_2 .

The necessary first – order conditions are:

$\partial\Omega / \partial e_1 = a_1 - e_2 - 2e_1 = 0$ ………………………. (7)

$\partial\Omega / \partial e_2 = a_2 - e_1 = 0$ …………………………… (8)

$\partial\Omega / \partial \lambda_1 = 2a_1 - a_2 - \beta = 0$ ………………………. (9)

$\partial\Omega / \partial \lambda_2 = 2a_2 - a_1 - \gamma = 0$ ………………………. (10)

From equation: (7) and (8) we have:

$2e_1 + e_2 = a_1$ and $e_1 = a_2 \rightarrow e_2 = a_1 - 2e_1 \rightarrow$ **Reaction function for neighbor: 2**

and $e_1 = (a_1 - e_2) / 2 \rightarrow$ **Reaction function for neighbor: 1**.

Again by the help of equations: (7) - (10) we derive the NE solution as:

$$(e_1^*, e_2^*) = (a_2, a_1 - 2a_2) = [(\beta + 2\gamma) / 3, - \gamma]$$

Similarly, the necessary Lagrangian equations for neighbor: 1 would be:

Maximize $\Phi = a_1 e_1 + a_2 e_2 - e_1 e_2 - e_2^2 + \lambda_1 [2a_1 - a_2 - \beta] + \lambda_2 [2a_2 - a_1 - \gamma]$
subject to e_1, e_2, λ_1 and λ_2 .

The necessary first – order conditions are:

$\partial\Phi / \partial e_1 = a_1 - e_2 = 0$ ………………………. (11)

$\partial\Phi / \partial e_2 = a_2 – e_1 - 2e_2 = 0$ ……………………… (12)

$\partial\Phi / \partial\lambda_1 = 2a_1 - a_2 - \beta = 0$ ……………………….. (13)

$\partial\Phi / \partial\lambda_2 = 2a_2 – a_1 – \gamma = 0$ ………………………. (14)

From equations: (11) and (12) we get:

$e_1 + 2e_2 = a_2$ and $e_2 = a_1 \rightarrow e_1 = a_2 – 2e_2 \rightarrow$ **Reaction function for neighbor: 1**

and $e_2 = (a_2 – e_1) / 2 \rightarrow$ **Reaction function for neighbor: 2**.

Again by the help of equations: (11) - (14) we derive the NE solution as:

$$(e_1^*, e_2^*) = (a_2 - 2a_1, a_1) = [-\beta, (2\beta + \gamma) / 3]$$

So the set of NE solutions would be:

$$e_1^* = [a_2, a_2 - 2a_1] = [(\beta + 2\gamma) / 3, - \beta];$$

$$e_2^* = [a_1 - 2a_2, a_1] = [-\gamma, (2\beta + \gamma) / 3].$$

(b) We already know that:

$U_1 = a_1 e_1 + a_2 e_2 - e_1 e_2 – e_1^2$ and $U_2 = a_1 e_1 + a_2 e_2 - e_1 e_2 – e_2^2$

Hence the payoff of 1st neighbor $(U_1) = a_1 e_1^* + a_2 e_2^* - e_1^* e_2^* – e_1^{*2}$

Now according to the problem, $a_1 = a_2 = 1$.

When $e_1^* = a_2$, then $e_2^* = a_1 - 2a_2 \rightarrow$ these values will help to rewrite the payoff equation of the 1st neighbor as:

$$U_1 = a_1 e_1^* + a_2 e_2^* - e_1^* e_2^* – e_1^{*2} = a_1 a_2 + a_2 (a_1 - 2a_2) - a_2 (a_1 - 2a_2) - a_2^2$$

$$= a_1 a_2 - a_2^2 = 1 – 1 = 0.$$

Now when $e_1^* = a_2 - 2a_1$, then $e_2^* = a_1$ – which gives us the payoff equation of the 1st neighbor as:

$$U_1 = a_1 e_1^* + a_2 e_2^* - e_1^* e_2^* – e_1^{*2} = a_1 (a_2 - 2a_1) + a_1 a_2 – a_1 (a_2 - 2a_1) – (a_2 - 2a_1)^2$$

$= -4.$

Now when $e_1^* = a_2$, then $e_2^* = (a_1 - 2a_2)$ – which gives us the payoff equation of the 2nd neighbor as:

$U_2 = a_1 e_1^* + a_2 e_2^* - e_1^* e_2^* - e_2^{*2} = a_1 a_2 + a_2 (a_1 - 2a_2) - a_2 (a_1 - 2a_2) - (a_1 - 2a_2)^2$

$= 0.$

Similarly, when $e_1^* = a_2 - 2a_1$, then $e_2^* = a_1$ – which gives us the payoff equation of the 1st neighbor as:

$U_1 = a_1 e_1^* + a_2 e_2^* - e_1^* e_2^* - e_2^{*2} = a_1 (a_2 - 2a_1) + a_1 a_2 - a_1 (a_2 - 2a_1) - a_1^2$

$= 0.$

Finally, when $e_1^* = a_2$, then $e_2^* = (a_1 - 2a_2)$ – which gives us the payoff equation of the 2nd neighbor as:

$U_2 = a_1 e_1^* + a_2 e_2^* - e_1^* e_2^* - e_2^{*2} = 0.$

However the Nash Equilibrium (NE) for pure strategy would give us the result:

$e_1^* = [a_2, a_2 - 2a_1] = [1, -1]$

$e_2^* = [a_1 - 2a_2, a_1] = [-1, 1]$ **when $a_1 = a_2 = 1$**.

(c) According to the problem, let $\bar{e}_1$ and $\bar{e}_2$ are the optimum levels of efforts for neighbor: 1 and 2 respectively which maximizes the combined effort $\bar{e} = \bar{e}_1 + \bar{e}_2$.

Now the quality of the road function is rewritten as:

$f(\bar{e}_1, \bar{e}_2) = a_1 \bar{e}_1 + a_2 \bar{e}_2 - \bar{e}_1 \bar{e}_2 \quad (a_1, a_2 > 0)$

and the cost of effort function would be: $C(\bar{e}_i) = (\bar{e}_1 + \bar{e}_2)^2$

Hence the required Lagrangian equation would be:

Maximize $l = a_1 \bar{e}_1 + a_2 \bar{e}_2 - \bar{e}_1 \bar{e}_2 - (\bar{e}_1 + \bar{e}_2)^2 + \lambda_1 [2a_1 - a_2 - \beta] + \lambda_2 [2a_2 - a_1 - \gamma]$

subject to $\bar{e}_1$, $\bar{e}_2$, λ_1 and λ_2 ($\lambda_1, \lambda_2, \beta, \gamma \geq 0$).

The required first – order conditions are:

$\partial l / \partial \bar{e}_1 = a_1 - 2\bar{e}_1 - 3\bar{e}_2 = 0$ ……………………… (I)

$\partial l / \partial \bar{e}_2 = a_2 - 3\bar{e}_1 - 2\bar{e}_2 = 0$ ……………………… (II)

$\partial l / \partial \lambda_1 = 2a_1 - a_2 - \beta = 0$ ………………………….. (III)

$\partial l / \partial \lambda_2 = 2a_2 - a_1 - \gamma = 0$ ………………………….. (IV)

Solving equations: (I) and (II) we get:

$\bar{e}_1 = (3a_2 - 2a_1) / 5$ and $\bar{e}_2 = (3a_1 - 2a_2) / 5$.

Now by putting $a_1 = a_2 = 1$ we get:

$\bar{e}_1 = \bar{e}_2 = 1 / 5$.

Thus the **maximized** combined effort would be: $\bar{e} = \bar{e}_1 + \bar{e}_2 = 2 / 5$.

Finally, **1st neighbor's payoff (U_1)** $= a_1 \bar{e}_1 + a_2 \bar{e}_2 - \bar{e}_1 \bar{e}_2 - \bar{e}_1^{\,2}$
$= (2 / 5) - (2 / 25) = 8 / 25$.

Similarly, **2nd neighbor's payoff (U_2)** $= a_1 \bar{e}_1 + a_2 \bar{e}_2 - \bar{e}_1 \bar{e}_2 - \bar{e}_2^{\,2}$
$= 8 / 25$.

(d) According to the problem, we have to rewrite the Lagragian equation with a *discount factor* (δ).

Now let the *optimum discounted* payoff functions for the 1st and 2nd neighbor be U_1^d and U_2^d respectively.

So we can write the discounted payoff functions as:

$U_1^d = a_1 e_1^* + a_2 e_2^* - e_1^* e_2^* - e_1^{*2} / 1 - \delta$ ……………………….. (i)

and $U_2^d = a_1 e_1^* + a_2 e_2^* - e_1^* e_2^* - e_2^{*2} / 1 - \delta$ ………………….. (ii) where (e_1^*, e_2^*) = $(\beta / 3, \gamma / 3) = [(2a_1 - a_2) / 3, (2a_2 – a_1) / 3]$ (derived form part : a)

[(e_1^*, e_2^*) correspond to pure strategy NE solution]

Thus substituting the values of e_1^* and e_2^* in equations: (i) and (ii) we get:

$U_1^d = [\ a_1. (2a_1 - a_2) / 3 + a_2. (2a_2 – a_1) / 3 - (2a_1 - a_2) (2a_2 – a_1) / 9 - (2a_1 - a_2)^2 /9] / 1 – \delta$

$= 4 / 9(1 – \delta)$ as by assumption $a_1 = a_2 = 1$.

Now according to the problem,

$4 / 9(1 – \delta) = 8 / 25 = 0.32$

→ δ = 0.28

Similarly, $U_2^d = 4 / 9(1 – \delta)$ and by proceeding likewise we find that **δ = 0.28 [$\delta \in (0, 1)$]**.

Hence **δ = 0.28** gives us the subgame perfect NE solution.